

# THE RIEMANN CONSTANT FOR A NON-SYMMETRIC WEIERSTRASS SEMIGROUP

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**ABSTRACT.** The zero divisor of the theta function of a compact Riemann surface  $X$  of genus  $g$  is the canonical theta divisor of  $\text{Pic}^{(g-1)}$  up to translation by the Riemann constant  $\Delta$  for a base point  $P$  of  $X$ . The complement of the Weierstrass gaps at the base point  $P$  given as a numerical semigroup plays an important role, which is called the Weierstrass semigroup. It is classically known that the Riemann constant  $\Delta$  is a half period  $\frac{1}{2}\Gamma_\tau$  for the Jacobi variety  $\mathcal{J}(X) = \mathbb{C}^g/\Gamma_\tau$  of  $X$  if and only if the Weierstrass semigroup at  $P$  is symmetric. In this article, we analyze the non-symmetric case. Using a semi-canonical divisor  $D_0$ , we show a relation between the Riemann constant  $\Delta$  and a half period  $\frac{1}{2}\Gamma_\tau$  of the non-symmetric case. We also identify the semi-canonical divisor  $D_0$  for trigonal curves, and remark on an algebraic expression for the Jacobi inversion problem using the relation.

**2000 MSC:** 14H55, 14H50, 14K25, 14H40

## 1. INTRODUCTION

The Riemann constant is an important invariant that is associated to a pointed curve  $(X, P)$ , whose Abel map is normalized at  $P$ . In this article we work over the complex numbers, assume that  $X$  is a compact Riemann surface of genus  $g > 1$  (which we call simply a curve), and use standard convention, in particular identifying divisors under addition with line bundles under multiplication, the Jacobian of  $X$  with the complex torus  $\mathcal{J}(X) := \mathbb{C}^g/\Gamma_\tau$ , where  $\Gamma_\tau$  is the period lattice associated with the choice of a standard homology basis  $\langle \alpha_i, \beta_i \rangle_{1 \leq i \leq g}$  on  $X$ . We call the corresponding basis of normalized holomorphic forms  $\omega_i$ ,  $1 \leq i \leq g$ , we denote by  $w$  the Abel map, by  $\kappa$  the natural projection  $\mathbb{C}^g \rightarrow \mathcal{J}(X)$ , and by  $\mathcal{K}_X$  the canonical divisor of  $X$ .

Despite the choices involved, there is a valuable uniqueness to the Riemann constant: it is the vector  $\Delta \in \mathbb{C}^g$  such that, for a given base point  $P$ ,  $\theta \left( \int_{gP}^{P_1+\dots+P_g} \omega - \Delta \right) = 0$  if and only if  $P_1+\dots+P_g-P$  is a positive divisor [F, Th. 1.1]; moreover,  $2\Delta+w(\mathcal{K}_X-(g-1)P) = 0$  modulo  $\Gamma_\tau$  [F, Le]. These facts connect the Riemann constant with the set of semi-canonical divisors, which in turn correspond to theta characteristics in  $\frac{1}{2}\mathbb{Z}^{2g}$ .

The complement of the Weierstrass gaps at  $P$  is called the Weierstrass semigroup if it is a numerical semigroup. We consider the pointed curve  $(X, P)$  associated with a Weierstrass semigroup.

When the curve  $X$  has the property that a canonical divisor  $\mathcal{K}_X = (2g - 2)P$ , or equivalently, the number  $(2g - 1)$  is (the last) Weierstrass gap, the Weierstrass semigroup at  $P$  is symmetric, i.e., an integer  $n$  belongs to it if and only if  $\ell_g - n$  does not, where  $1 < \ell_1 < \ell_2 < \dots < \ell_g$  are the Weierstrass gaps at  $P$  [S]. This is also the case if and only if the Riemann constant is a half period,  $\Delta \in \frac{1}{2}\Gamma_\tau$ : the parity and addition theorems for the theta function become simpler than in general [F]. It is only in this case that, to the best of our knowledge, the Riemann constant has been written explicitly; in the hyperelliptic case, it is  $\kappa\Delta = \tau[\frac{1}{2}, \dots, \frac{1}{2}] + [\frac{g}{2}, \frac{g-1}{2}, \dots, \frac{1}{2}]$  [M2, IIIa (5.4)] and in the trigonal cyclic case of the Picard curve, it is  $\kappa\Delta = \tau[0 \ \frac{1}{2} \ 0] + [0 \ \frac{1}{2} \ 0]$  [Sh, Prop. I-2].

The Weierstrass semigroup at  $P$  may fail to be symmetric, although the problem of finding all numerical semigroups that can be realized on a pointed curve is still open (cf. [Pi]). As far as we know, the results in [MK, KMP] about the Jacobi inversion problem are the only ones given so far for the non-symmetric case. We were able to generalize results of Mumford's [M2] on hyperelliptic functions, in particular for the semigroups  $\langle 3, 4, 5 \rangle$ , and  $\langle 3, 7, 8 \rangle$ .

In this article, we state the algebraic-transcendental correspondence for the Riemann constant on a general pointed curve and its consequences for the Jacobi inversion problem. In the recent monograph [FZ], the authors analyze the Riemann constant as well as we do, for  $Z_n$  curves, which have a total ramification at  $P$ ; their goal is to derive (explicitly in some cases) an analog of the classical Thomae formula, which expresses algebraically the branch points of the curve in terms of theta characteristics and thetanulls. Our motivation instead is to use the “shifted Riemann constant”  $\Delta_s$ , cf. Definition 4.4, and “shifted Abel map”  $w_s$ , cf. Definition 4.7, to make explicit the correspondence between the group structure of the Jacobian and linear equivalence of divisors on the curve, a crucial issue when evaluating the sigma function (associated to the theta function) on symmetric products of the curve, cf. Section 4. More precisely we introduce a divisor  $\mathcal{B}$  related to a semi-canonical divisor  $D_0$  in Lemma 3.1, and using  $\mathcal{B}$ , we define the shifted Riemann constant  $\Delta_s$  and shifted Abel map  $w_s$  to connect the shifted Riemann constant  $\Delta_s$  with a half period  $\frac{1}{2}\Gamma_\tau$ . The main results are given in Theorems 4.5, 4.8 and 4.9. The case when  $P$  is a trigonal point is made explicit in Proposition 3.3. As in Remark 4.11 and (4.1), we show that they play crucial roles in the Jacobi inversion problem.

After setting up notation and standard facts in Section 2, we explain our construction of the non-symmetric examples in Section 3 and interpret them in the classical setting of theta characteristics in Section 4.

**Acknowledgments:** One of the authors (S.M.) thanks Atsushi Nakayashiki for pointing out this problem for [KMP, MK], and Yoshihiro Ônishi for critical discussions.

## 2. NOTATION AND REVIEW

**2.1. Abel map and theta functions.** Let us consider a compact Riemann surface  $X$  of genus  $g$  and its Jacobian  $\mathcal{J}(X) := \mathbb{C}^g / \Gamma_\tau$  where  $\Gamma_\tau := \mathbb{Z}^g + \tau\mathbb{Z}^g$ . Let  $\hat{X}$  be an Abelian

covering of  $X$  ( $\varpi : \hat{X} \rightarrow X$ ). Since the covering space  $\hat{X}$  is constructed by a quotient space of path space (contour of integral), we consider an embedding  $X$  into  $\hat{X}$  by a map  $\iota : X \rightarrow \hat{X}$  such that  $\varpi \circ \iota = id$ . For a point  $P \in \hat{X}$ , we define the Abel map  $w$  and  $\tilde{w}$

$$w : S^k \hat{X} \rightarrow \mathbb{C}^g, \quad w(P_1, \dots, P_k) = \sum_{i=1}^k w(P_i) = \sum_{i=1}^k \int_P^{P_i} \omega, \quad \tilde{w} := \kappa w \iota : S^k X \rightarrow \mathcal{J}(X),$$

where  $S^k \hat{X}$  and  $S^k X$  are  $k$ -symmetric products of  $\hat{X}$  and  $X$  respectively. The Abel theorem shows  $\kappa w = \tilde{w} \varpi$ . We fix  $\iota$  and  $\varpi P$  is also denoted by  $P$ .

Later we will fix the point  $P \in X$  as a marked point  $P$  of  $(X, P)$ .

The map  $\tilde{w}$  embeds  $X$  into  $\mathcal{J}(X)$  and generalizes to a map from the space of divisors of  $X$  into  $\mathcal{J}(X)$  as  $\tilde{w}(\sum_i n_i P_i) := \sum_i n_i \tilde{w}(P_i)$ ,  $P_i \in X$ ,  $n_i \in \mathbb{Z}$ . Similarly we will define  $w(\iota D)$  for a divisor  $D$  of  $X$ . (In Introduction, we omitted  $\iota$  and wrote  $w(D)$  rather than  $w(\iota D)$ .)

The Riemann theta function, analytic in both variables  $z$  and  $\tau$ , is defined by:

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp \left( 2\pi\sqrt{-1} \left( {}^t n z + \frac{1}{2} {}^t n \tau n \right) \right).$$

The zero-divisor of  $\theta$  modulo  $\Gamma_\tau$  is denoted by  $\Theta := \kappa \text{div}(\theta) \subset \mathcal{J}(X)$ .

The theta function with characteristics  $\delta', \delta'' \in \mathbb{R}^g$  is defined as:

$$(2.1) \quad \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp \left[ \pi\sqrt{-1} \left\{ {}^t (n + \delta') \tau (n + \delta') + 2 {}^t (n + \delta') (z + \delta'') \right\} \right].$$

If  $\delta = (\delta', \delta'') \in \{0, \frac{1}{2}\}^{2g}$ , then  $\theta[\delta](z, \tau) := \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (z, \tau)$  has definite parity in  $z$ ,  $\theta[\delta](-z, \tau) = e(\delta) \theta[\delta](z, \tau)$ , where  $e(\delta) := e^{4\pi i \delta'^t \delta''}$ . There are  $2^{2g}$  different characteristics of definite parity.

**2.2. Numerical semigroups.** A numerical semigroup  $H = \langle M \rangle$  generated by a set  $M$ , has gap sequence  $L(H) := \mathbb{N}_0 \setminus H$  of  $H$  and a finite number  $g(H)$ , called “genus”, of elements in  $L(H)$ . For example,

$$L(\langle 3, 4, 5 \rangle) = \{1, 2\}, \quad L(\langle 3, 7, 8 \rangle) = \{1, 2, 4, 5\}.$$

For a gap sequence  $L := \{\ell_0 < \ell_1 < \dots < \ell_{g-1}\}$  of genus  $g$ , let  $M(L)$  be the minimal set of generators for the semigroup  $H(L)$  and

$$(2.2) \quad \alpha(L) := \{\alpha_0(L), \alpha_1(L), \dots, \alpha_{g-1}(L)\}$$

where  $\alpha_i := \ell_i - i - 1$ . When an  $\alpha_i$  is repeated  $j > 1$  times we write  $\alpha_i^j$  in  $\alpha(L)$ . We let:  $\lambda_i(L) := \alpha_{g-i}(L) + 1$  and associate to  $L$  the Young diagram  $\Lambda(L) = (\lambda_1(L), \dots, \lambda_g(L))$ .

We say that  $H$  is symmetric when it has a property that an integer  $n$  belongs to it if and only if  $\ell_g - n$  does not. Therefore,  $H$  is symmetric if and only if  $\Lambda(L(M))$  is self-dual; in this case, the Young diagram is equal to its transpose. It is known that  $H$  is symmetric if and only if  $2g(H) - 1$  is a gap of  $H$ .

We let  $a_{\min}(L)$  be the smallest positive integer of  $M(L)$ . We call a semigroup  $H$  an  $a_{\min}(L)$ -semigroup; for example,  $\langle 3, 4, 5 \rangle$  and  $\langle 3, 7, 8 \rangle$  are 3-semigroups.

For a curve  $X$  of genus  $g$  and a point  $P \in X$ , the semigroup

$$H(X, P) := \{n \in \mathbb{N}_0 \mid \text{there exists } f \in k(X) \text{ such that } (f)_\infty = nP\},$$

called the Weierstrass semigroup of the point  $P$ , is a numerical semigroup and  $g = g(H(X, P))$ . If  $L(H(X, P)) := \mathbb{N}_0 \setminus H(X, P)$  differs from the set  $\{1, 2, \dots, g\}$ , we say that  $P$  is a Weierstrass point of  $X$ . We also say that the pointed curve  $(X, P)$  is symmetric (non-symmetric) if such is the Weierstrass semigroup at  $P$ .

### 3. CANONICAL DIVISOR OF A POINTED CURVE

In this section we give a lemma on the canonical divisors of a pointed curve  $(X, P)$  with Weierstrass semigroup  $H(X, P)$ , including non-symmetric ones. The fact is classical, cf., e.g., [Sh, Th. A-3], [M1, Definition 3.9, p.163].

**Lemma 3.1.** *There is a divisor  $\mathcal{B}$  of degree  $d_0 := \deg(\mathcal{B})$  such that*

$$\mathcal{K}_X = 2D_0 = 2(g - 1 + d_0)P - 2\mathcal{B},$$

where we indicate linear equivalence by an equal sign.

*Proof.* By the surjectivity of the Abel map, there exists a divisor  $D_0$  such that  $\mathcal{K}_X = 2D_0$ . ( $D_0$  is called the semi-canonical divisor.) Since the degree of  $\mathcal{K}_X$  is  $2g - 2$ , there is a divisor  $\mathcal{B}$  such that  $D_0 = (g - 1 + d_0)P - \mathcal{B}$ .  $\square$

**Remark 3.2.** (1) As we show later, the divisor  $\mathcal{B}$  plays much more important roles than the semi-canonical divisor  $D_0$  in this article, though  $\mathcal{B}$  is defined from  $D_0$ .

(2) If  $H(X, P)$  is symmetric,  $\mathcal{B}$  can be taken to be the zero divisor and  $d_0 = 0$ , therefore Lemma 3.1 includes the symmetric Weierstrass semigroup at  $P$  as a special case.

(3) A pointed curve hyperelliptic at  $P$  (namely,  $H(X, P)$  contains the number 2), is symmetric. If the Weierstrass semigroup of the curve is generated by two numbers  $\langle r, s \rangle$ , the semigroup is symmetric. For example, a trigonal non-singular plane curve  $y^3 = x^r + 1$  is a  $\langle 3, r \rangle$ -type symmetric curve (at  $P = \infty$ ). Since a trigonal curve with total ramification at  $P$  has semigroup generated by either two or three elements, this remark and Proposition 3.3 cover all trigonal cases with total ramification.

(4) The semi-canonical divisor  $D_0$  i.e.,  $2D_0 = \mathcal{K}_X$  is sometimes called *theta characteristics* [M1, Definition 3.9, p.163]. Instead of this terminology, we refer to the vector  $\delta$  in (2.1) as the *theta characteristics*.

**Proposition 3.3.** *For a pointed curve  $(X, P)$  whose Weierstrass semigroup at  $P$  is of type  $\langle 3, 2r + s, 2s + r \rangle$  for natural numbers  $(r, s) \neq 1$ ,  $r > s$  we have the following results:*

(1)  $(X, P)$  is not symmetric.

(2) The genus of  $X$  is  $g = r + s - 1$ ; and

(3) the divisor  $\mathcal{B}$  in Lemma 3.1 can be written explicitly as  $\mathcal{B} = B_{s+1} + \cdots + B_{s+r}$ , with

$$\mathcal{K}_X = (2g - 2)P + B_1 + \cdots + B_s - sP,$$

where  $B_1, \dots, B_{r+s}$  are ramification points corresponding to the branch points of a singular curve,

$$f_0(x, y) = y^3 - (x - b_1) \cdots (x - b_s) \cdot (x - b_{s+1})^2 \cdots (x - b_{s+r})^2 = 0,$$

with  $P = \infty$ ,

$$B_1 + \cdots + B_s + 2(B_{s+1} + \cdots + B_{s+r}) - (s + 2r)P \sim 0,$$

but

$$B_{s+1} + \cdots + B_{s+r} - rP \not\sim 0.$$

*Proof.* As indicated in [MK, KMP], where the result was proven for the cases  $\langle 3, 4, 5 \rangle$  and  $\langle 3, 7, 8 \rangle$  corresponding to  $(r = 1, s = 2)$  and  $(r = 2, s = 3)$  respectively, the result holds in general. More precisely, by normalizing the singular curve, we also have [MK],

$$f_1(x, y) = w^3 - (x - b_1)^2 \cdots (x - b_s)^2 \cdot (x - b_{s+1}) \cdots (x - b_{s+r}) = 0.$$

In other words, we have the commutative ring  $R = \mathbb{C}[x, y, w]/(h_1, h_2, h_3)$  of the curve  $(X, P)$ ,  $R = \mathcal{O}_X(*P)$ , where

$$\begin{aligned} h_1(x, y) &= w^2 - (x - b_1) \cdots (x - b_s)y = 0, \\ h_2(x, y) &= y^2 - (x - b_{s+1}) \cdots (x - b_{s+r})w = 0, \\ h_3(x, y) &= wy - (x - b_1) \cdots (x - b_{s+r}) = 0, \end{aligned}$$

By simple computations, we have

$$\left( \frac{1}{w} dx \right) = (B_1 + \cdots + B_s) + (2g - 2 - s)P.$$

□

**Remark 3.4.** We give some comments on the pointed curve  $(X, P)$  of type  $\langle 3, 2r+s, 2s+r \rangle$  treated in Proposition 3.3. The curve  $\langle 3, 7, 8 \rangle$  and the related cyclic singular curves are also studied in [FZ, p. 83] (without consideration of the Weierstrass semigroup), for the generalizations of Thomae's formula. In this article, we consider these singular curves for the Jacobi inversion formulae, so we focus on the affine ring  $\text{Spec} R$  by normalizing these curves. This allows us to use linear-equivalence relations, such as  $3B_i \sim 0$  and

$$\begin{aligned} B_1 + \cdots + B_s + B_{s+1} + \cdots + B_{s+r} - (s + r)P &\sim -(B_{s+1} + \cdots + B_{s+r} - rP) \\ &\sim 2(B_{s+1} + \cdots + B_{s+r} - rP). \end{aligned}$$

By letting  $\mathcal{B}_1 := B_1 + \cdots + B_{r+s}$ ,  $\mathcal{B}_1 - (s + r)P \sim 2(\mathcal{B} - rP)$ . Further we have

$$\left( \frac{1}{wy} dx \right) = -\mathcal{B}_1 + (2g - 2 + r + s)P.$$

We define  $R^B := \{f \in R \mid \exists \ell, \text{ such that } (f) - \mathcal{B}_1 + \ell P > 0\}$ , so that  $y$  and  $w$  belong to  $R^B$ . We choose a graded (by order of pole) basis of  $R^B = \oplus_{i=0} \mathbb{C}f_i$  as a  $\mathbb{C}$ -vector space. Then the holomorphic one-forms are explicitly written as

$$\frac{f_i}{wy} dx \quad i = 0, \dots, g-1.$$

By Riemann-Roch theorem, the degree of  $f_{g-2}$  at  $P$  is  $(2g-2) + (r+s)$ .

We display the examples of  $R$  and  $R^B$ :

Table 1: Examples of  $R$

$(r, s)$	$g \setminus \text{wt}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
(1, 3)	3	1	-	-	$x$	-	$w$	$x^2$	$y$	$xw$	$x^3$	$xy$	$x^2w$	$wy$	$x^2y$	$x^3w$	$xwy$	$x^3y$	$w^2y$	$x^2wy$
(2, 3)	4	1	-	-	$x$	-	-	$x^2$	$w$	$y$	$x^3$	$xw$	$xy$	$x^4$	$x^2w$	$x^2y$	$wy$	$x^3w$	$x^3y$	$xwy$
(1, 5)	5	1	-	-	$x$	-	-	$x^2$	$w$	-	$x^3$	$xw$	$y$	$x^4$	$x^2w$	$xy$	$x^5$	$x^3w$	$x^2y$	$wy$
(2, 4)	5	1	-	-	$x$	-	-	$x^2$	-	$w$	$x^3$	$y$	$xw$	$x^4$	$xy$	$x^2w$	$x^5$	$x^2y$	$x^3w$	$wy$
(3, 4)	6	1	-	-	$x$	-	-	$x^2$	-	-	$x^3$	$w$	$y$	$x^4$	$xw$	$xy$	$x^5$	$x^2w$	$x^2y$	$x^6$

Table 2: Examples of  $R^B$

$(r, s)$	$g \setminus \text{wt}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
(1, 3)	3	-	-	-	-	-	$w$	-	$y$	$xw$	-	$xy$	$x^2w$	$wy$	$x^2y$	$x^3w$	$xwy$	$x^3y$	$w^2y$	$x^2wy$
(2, 3)	4	-	-	-	-	-	-	$w$	$y$	-	$xw$	$xy$	-	$x^2w$	$x^2y$	$wy$	$x^3w$	$x^3y$	$xwy$	
(1, 5)	5	-	-	-	-	-	-	$w$	-	-	$xw$	$y$	-	$x^2w$	$xy$	-	$x^3w$	$x^2y$	$wy$	
(2, 4)	5	-	-	-	-	-	-	-	$w$	-	$y$	$xw$	-	$xy$	$x^2w$	-	$x^2y$	$x^3w$	$wy$	
(3, 4)	6	-	-	-	-	-	-	-	-	-	$w$	$y$	-	$xw$	$xy$	-	$x^2w$	$x^2y$	-	

#### 4. RIEMANN CONSTANT

From Theorem 7 in [Le], we have

**Proposition 4.1.**

$$\tilde{w}(\mathcal{S}^{g-1}X) + \kappa\Delta = \Theta \quad \text{modulo } \Gamma_\tau$$

by letting the Riemann constant denoted by  $\Delta \in \mathbb{C}^g$ .

It implies that, up to translation by the vector  $\Delta$ , the zero-divisor  $\Theta$  of  $\theta$  is the canonical theta divisor of  $\text{Pic}^{(g-1)}$ .

Theorem 11 in [Le] says:

**Proposition 4.2.** *An effective divisor  $D$  whose degree is  $2g-2$  satisfies  $\tilde{w}(D - (2g-2)P) + 2\kappa\Delta = 0$  modulo  $\Gamma_\tau$  if and only if  $D$  is the divisor of the holomorphic one form, i.e.,  $\tilde{w}(\mathcal{K}_X - (2g-2)P) + 2\kappa\Delta = 0$  modulo  $\Gamma_\tau$ .*

As mentioned in the Introduction, the following result is classical (e.g, [Sh, Appendix, Cor. 2]):

**Lemma 4.3.** *The Riemann vector  $\Delta$  belongs to  $\frac{1}{2}\Gamma_\tau$  if and only if  $\mathcal{K}_X = (2g - 2)P$ .*

**Definition 4.4.** *We define the “shifted Riemann constant” by a translation:*

$$\Delta_s := \Delta - w(\iota\mathcal{B}) \in \mathbb{C}^g.$$

**Theorem 4.5.**  *$\Delta_s$  belongs to  $\frac{1}{2}\Gamma_\tau$ .*

*Proof.* From Proposition 4.2 and Lemma 3.1,

$$\tilde{w}(\mathcal{K}_X - (2g - 2)P) + 2\kappa\Delta = \tilde{w}(\mathcal{K}_X - (2g - 2)P + 2\mathcal{B}) + 2\kappa\Delta_s = 0 \quad \text{modulo } \Gamma_\tau.$$

On the other hand, the first term vanishes, because

$$\tilde{w}(\mathcal{K}_X - (2g - 2)P + 2\mathcal{B}) = \tilde{w}(0) = 0 \quad \text{modulo } \Gamma_\tau.$$

Hence  $2\Delta_s = 0$  modulo  $\Gamma_\tau$  and the statement is proved. We note that this property does not depend on the choice of the identification  $\iota$ .  $\square$

**Remark 4.6.** Theorem 4.5 gives a correspondence between the Riemann constant of a general pointed curve and a half-period point of the Jacobian.

There are  $2^{2g}$  elements of  $(\frac{1}{2}\Gamma_\tau)/\Gamma_\tau$ , which is bijective to a set  $\Sigma$  whose elements  $D$  are defined by  $2D = \mathcal{K}_X$  [M1, p.163], [Sh, Appendix]. Therefore, the choice of  $D_0$  in Lemma 3.1 typically is not explicit, rather, the Lemma is an existence statement. For a curve as in Proposition 3.3,  $\mathcal{B}$  is naturally determined.

Our construction is very similar to the relation between the half-period and  $D - D_0$  of  $D, D_0 \in \Sigma$  [Sh, Th. A-4]. However, as in Remark 3.2 (1), we use different divisors; especially,  $\mathcal{B}$  does not belong to  $\Sigma$  in general.

**Definition 4.7.** *The “shifted Abel map” is defined by*

$$w_s(P_1, \dots, P_k) = w(P_1, \dots, P_k) + w(\iota\mathcal{B}),$$

and  $\tilde{w}_s := \kappa w_s$ . In other words it is given by translating a divisor  $D$ ,  $w_s(\iota D) = w(\iota(D + \mathcal{B}))$  and  $\tilde{w}_s(D) = \tilde{w}(D + \mathcal{B})$ .

**Theorem 4.8.**  *$\tilde{w}_s(P_1, \dots, P_{g-1}) + \kappa\Delta_s$  is in the theta divisor  $\Theta$  modulo  $\Gamma_\tau$ ; or more explicitly,*

$$\tilde{w}_s(\mathcal{S}^{g-1}X) + \kappa\Delta_s = \Theta \quad \text{modulo } \Gamma_\tau$$

*using the shifted Abel map.*

It also implies that, up to translation by the vector  $\Delta_s$ , the zero-divisor  $\Theta$  of  $\theta$  is the canonical theta divisor of  $\text{Pic}^{(g-1)}$  using the shifted Abel map.

*Proof.*  $\tilde{w}_s(P_1, \dots, P_{g-1}) + \kappa\Delta_s = \tilde{w}(P_1, \dots, P_{g-1}) + \kappa\Delta_s + \tilde{w}(\mathcal{B}) = \tilde{w}(P_1, \dots, P_{g-1}) + \kappa\Delta$ . Proposition 4.1 means that the left hand side of the formula is  $\Theta$ .  $\square$

In the paper [KMP], we call  $\Delta_s$  itself *the Riemann constant* since it plays such role under the shifted Abel map.

**Theorem 4.9.** *There exists a theta characteristic  $\delta = \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \in \left\{0, \frac{1}{2}\right\}^{2g}$  so that*

$$\theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (w_s(P_1, \dots, P_{g-1})) = 0$$

for every  $P_i \in \hat{X}$ , i.e., for  $\Theta_s := \kappa \operatorname{div}(\theta[\delta])$ ,  $\Theta_s = \tilde{w}_s(S^{g-1}X)$ .

**Remark 4.10.** As in [KMP], our goal is to extend the classical knowledge of Jacobi's inversion problem, where the theta function with characteristics  $\delta \in \{0, \frac{1}{2}\}^{2g}$  plays a central role. Ultimately, Jacobi's inversion connects the meromorphic and the Abelian functions of a curve. Theorem 4.9 enables us to use the properties of the theta function with theta characteristics  $\delta \in \{0, \frac{1}{2}\}^{2g}$  for non-symmetric pointed curves as well.

More precisely, our previous work was concerned with the “sigma function”, a generalization of Weierstrass' elliptic sigma function; the higher-genus case sigma is, up to an exponential multiplicative factor quadratic in the exponent, the theta function with theta characteristics  $\delta \in \{0, \frac{1}{2}\}^{2g}$ .

In fact, in [MK, KMP], we defined the sigma function and solved the Jacobi inversion problem, but although we used the above Theorems 4.5, 4.8 and 4.9 implicitly, we defined the shifted Riemann constant by checking *ad hoc* calculations. In this article, we streamline the theory, which is based on simple group-theoretic properties of the Jacobian, and have a geometric meaning given in the following Remark.

**Remark 4.11.** Assume that  $\mathcal{B}_1$  is an effective divisor such that  $d_1 := \deg(\mathcal{B}_1) < 2d_0$  and  $\mathcal{B}_1 - d_1P \sim 2\mathcal{B} - 2d_0P$  as in Proposition 3.3 and Remark 3.4. Let us define  $R = \mathcal{O}_X(*P)$ , the ring of meromorphic functions with pole at most at  $P$ , and  $R^B := \{f \in R \mid \exists \ell, \text{ such that } (f) - \mathcal{B}_1 + \ell P > 0\}$ . Thus,  $R^B$  is a natural commutative ring that keeps track of the Weierstrass semigroup of a pointed curve.

We choose a graded basis of  $R^B = \bigoplus_{i=0} \mathbb{C}f_i$  as a  $\mathbb{C}$ -vector space arising from the elements of  $H(X, P)$ . Let  $n$  be a positive integer and  $P_1, \dots, P_n$  be in  $X \setminus P$ . We define the *Frobenius-Stickelberger (FS) determinant* by

$$\psi_n(P_1, P_2, \dots, P_n) := \begin{vmatrix} f_0(P_1) & f_1(P_1) & \cdots & f_{n-1}(P_1) \\ f_0(P_2) & f_1(P_2) & \cdots & f_{n-1}(P_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(P_n) & f_1(P_n) & \cdots & f_{n-1}(P_n) \end{vmatrix}$$

and  $\mu$ -functions by

$$\mu_n(P) := \mu_n(P; P_1, \dots, P_n) := \lim_{P'_i \rightarrow P_i} \frac{1}{\psi_n(P'_1, \dots, P'_n)} \psi_{n+1}(P'_1, \dots, P'_n, P),$$



where the  $P'_i$  are generic, the limit is taken (irrespective of the order) for each  $i$ . Here we also have  $\mu_{n,k}(P_1, \dots, P_n)$  by

$$\mu_n(P) = f_n(P) + \sum_{k=0}^{n-1} (-1)^{n-k} \mu_{n,k}(P_1, \dots, P_n) f_k(P),$$

with the convention  $\mu_{n,n}(P_1, \dots, P_n) \equiv 1$ . The divisor  $(\mu_n(P))$  shows that there are points  $Q_i \in X$  such that

$$\sum_{i=1}^n P_i + \sum_{i=1}^{N(n)-n-2d_0} Q_i - N(n)P + \mathcal{B}_1 \sim 0$$

where  $N(n) \in H(X, P)$  is the order of  $\mu_n$ . Noting the definition of  $\mathcal{B}_1$  This implies that the addition structure of the shifted Abelian map satisfies:

$$\left( \sum_{i=1}^n P_i + \mathcal{B} \right) - (n + d_0)P \sim - \left( \sum_{i=1}^{N(n)-n-d_1} Q_i + \mathcal{B} \right) - (N(n) - n - d_1 + d_0)P.$$

For the  $n = g - 1$  case, by assuming the fact that  $N(g - 1) = 2g - 2 + d_1$  as in Remark 3.4, we have

$$\left( \sum_{i=1}^{g-1} P_i + \mathcal{B} \right) - (g - 1 + d_0)P \sim - \left( \sum_{i=1}^{g-1} Q_i + \mathcal{B} \right) - (g - 1 + d_0)P.$$

This corresponds to a symmetry of  $\Theta_s$  in Theorem 4.9 under the minus operation on the Jacobian as in [M1, p.166], which makes the shifted Abel map natural, i.e.,

$$(4.1) \quad \Theta_s = -\Theta_s, \quad \tilde{w}_s(S^{g-1}X) = -\tilde{w}_s(S^{g-1}X).$$

Further by the explicit Jacobi inversion formulae, we connect the theta function with half-integer theta characteristics and  $\mu_{n,k}$ , cf. [KMP, MK].

**Remark 4.12.** As mentioned in the Introduction, we introduced the “shifted Riemann constant” in Definition 4.4 and “shifted Abel map” Definition 4.7 which enable us to handle theta function with theta characteristics in Theorem 4.9. Our results hold for general pointed curves with Weierstrass semigroup generated by a lowest integer  $n$ , not necessarily  $Z_n$  curves as in [FZ]. Our main motivation is an explicit expression of linear equivalence of divisors in terms of the addition structure of the Jacobian as a complex torus.

## REFERENCES

- [F] J.D. Fay, Theta functions on Riemann Surfaces, Lecture Notes in Mathematics, Vol. 352. Springer-Verlag, Berlin-New York, 1973.
- [H] J. Herzog, *Generators and relations of Abelian semigroup and semigroup ring*, Manu. Math., **3** (1970) 175-193.
- [Le] J. Lewittes, *Riemann surfaces and the theta functions*, Acta Math., **111** (1964) 37-61.

- [FZ] H. M. Farkas, S. Zemel, Generalizations of Thomae's Formula for  $Z_n$  Curves (Developments in Mathematics), Springer, New York, 2010.
- [KMP] J. Komeda, S. Matsutani and E. Previato, *The sigma function for Weierstrass semigroup  $\langle 3, 7, 8 \rangle$  and  $\langle 6, 13, 14, 15, 16 \rangle$* , Int. J. Math., **24** (2013) 1350085 (58 pages).
- [MK] S. Matsutani and J. Komeda, *Sigma functions for a space curve of type  $(3, 4, 5)$* , J. Geom. Symm. Phys., **30** (2013) 75-91, arXiv:1112.4137.
- [M1] D. Mumford, Tata Lectures on Theta I, Progress in Mathematics, 28. Birkhäuser, Boston, 1983.
- [M2] D. Mumford, Tata Lectures on Theta II, Progress in Mathematics, 43. Birkhäuser, Boston, 1984.
- [Pi] H. C. Pinkham, *Deformation of algebraic varieties with  $G_m$  action*, Astérisque, **20** (1974) 1-131.
- [Sh] H. Shiga, On the representation of the Picard modular function by  $\theta$  constants I-II, *Publ. RIMS, Kyoto Univ.*, 24:311-360, 1988.
- [S] Stöhr, Karl-Otto, On the moduli spaces of Gorenstein curves with symmetric Weierstrass semigroups. J. Reine Angew. Math. 441 (1993), 189-213.

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